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Polynomial rings of the chiral $SU(N)_2$ models

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Abstract. Via explicit diagonalization of the chiral $SU(N)_2$ fusion matrices, we discuss the possibility of representing the fusion ring of the chiral $SU(N)$ models, at level $K = 2$, by a polynomial ring in a single variable when N is odd and by a polynomial ring in two variables when N is even.

1. Introduction

Six years ago, Gepner conjectured that the fusion ring of theories with the $SU(N)$ current algebra is isomorphic to a ring in $N - 1$ variables associated with the fundamental representations, quotiented by an ideal of constraints that derive from a potential [1].

Four years ago, Di Francesco and Zuber postulated a necessary and sufficient condition for a one-variable polynomial ring [2]: assume that among the matrices N_i, $i = 1, \ldots, n$, there exists at least one, call it N_f, with non-degenerate eigenvalues. Thus, any other N_i may be diagonalized in the same basis as N_f and there exists a unique polynomial $P_i(x)$ of degree at most $n - 1$ such that its eigenvalues $\gamma_i^{(l)}$ satisfy

$$\gamma_i^{(l)} = P_i(\gamma_f^{(l)}) \tag{1}$$

P_i being given by the Lagrange interpolation formula. Therefore, any N_i may be written as

$$N_i = P_i(N_f) \tag{2}$$

with a polynomial P_i; as both N_i and N_f have integral entries, $P_i(x)$ must have rational coefficients.

The $n \times n$ matrix N_f, on the other hand, satisfies its characteristic equation $\mathcal{P}(x) = 0$, which is also its minimal equation, as N_f has no degenerate eigenvalues. The constraint on N_f is thus

$$\mathcal{P}(N_f) = 0 \tag{3}$$

that may of course be integrated to yield a 'potential' $\mathcal{W}(x)$, which is a polynomial of degree $n + 1$. In this way, Di Francesco and Zuber characterized the rational conformal field theories (RCFTs) which have a description in terms of a fusion potential in one variable. Moreover, they have also proposed a generalized potential to describe other theories. In [3] Aharony determined a simple criterion for a generalized description of RCFTs by fusion potentials in more than one variable.

In this paper we tackle this problem and discuss the possibility of representing the fusion rings of the chiral $SU(N)_2$ models by polynomial rings in two variables. Exploiting the

Di Francesco and Zuber condition, we show that these polynomial rings in two variables are reduced to polynomial rings in a single variable in the cases for which N is odd (or $N = 2$).

In section 2 we discuss some algebraic setting of the chiral RCFTs. Section 3 describes the primary fields of the chiral $SU(N)$ models, at level $K = 2$, in *cominimal equivalence classes*. In section 4 we report a computer study which diagonalizes the fusion matrices of the chiral $SU(N)_2$ models and gives their polynomial rings in one and two variables.

2. Fusion algebras

Fusion algebras are found to play an important role in the study of RCFTs. In addition, the fusion rules can be expressed in terms of the unitary matrix S [4] which encodes the modular transformations of the characters of the RCFT

$$N_{ij}^{k} = \sum_{l} \frac{S_{il}}{S_{0l}} \, S_{jl} S_{kl}^{*}. \tag{4}$$

Here '0' refers to the identity operator, and the labels $i, \ldots, l$ run over n values corresponding to the primary fields of the chiral algebra of the RCFT. There is a more fundamental reason for seeking representations of the fusion algebra, based on the concept of operator products [6]. When one tries to compute the operator product coefficients, one is almost inevitably led to the concept of fusion rules, i.e. the formal products

$$A_i A_j = \sum_{k} N_{ij}^{k} A_k \tag{5}$$

of primary fields describing the basis-independent content of the operator product algebra.

By definition, the fusion rule coefficients possess the property of integrality $N_{ij}^{k} \in Z_{\geqslant 0}$. In addition, they inherit several simple properties:

- *symmetry*: $N_{ij}^{k} = N_{ji}^{k}$,
- *associativity*: $\sum_k N_{ij}^{k} N_{kl}^{m} = \sum_k N_{jl}^{k} N_{ik}^{m}$,
- *existence of unit*: there is an index '0' (identity operator) such that $N_{i0}^{j} = \delta_i^j$, and
- *charge conjugation*: $N_{ijl} = \sum_k N_{ij}^{k} C_{kl} = (N_{ij}^{l})^{\dagger}$ is completely symmetric in the indices i, j, l.

Because of these properties, one can interpret the fusion rule coefficients as the structure constants of a commutative associative ring with a basis given by the primary fields.

The matrix S implements the modular transformation $\tau \to -1/\tau$ and obeys $S^2 = C$. In addition, the diagonal matrix $T_{ii} = \exp(2\mathrm{i}\pi(\Delta_i - c/24))$, where Δ_i is the conformal dimension of the primary field i and c is the central charge, implements the modular transformation $\tau \to \tau + 1$ and obeys $(ST)^3 = C$, which implies a relation between the structure constants N_{ij}^{k} and the conformal dimensions Δ_i [7]:

$$N_{ijkl}(\Delta_i + \Delta_j + \Delta_k + \Delta_l) = \sum_{r} N_{ijklr} \Delta_r \tag{6}$$

where

$$N_{ijkl} = N_{ij}^{\bar{n}} N_{kl}^{n} \qquad \text{and} \qquad N_{ijklr} = N_{ij}^{r} N_{klr} + N_{jk}^{r} N_{ilr} + N_{ik}^{r} N_{jlr}. \tag{7}$$

It was suggested in [8] that these proprieties fully characterize a RCFT, and that any commutative ring satisfying these properties is the fusion ring of some RCFT.

The matrices N_i defined by $(N_i)_{jk} = N_{ij}^k$ themselves form a trivial representation of the fusion algebra

$$N_i N_j = \sum_k N_{ij}^k N_k \tag{8}$$

as follows from unitarity of the matrix S; this expresses the associativity property of the algebra (5). The relation (4) implies that the matrix S diagonalizes the matrices N_i and their eigenvalues are of the form

$$\gamma_i^{(l)} = \frac{S_{il}}{S_{0l}} \tag{9}$$

and obey the sum rules

$$\gamma_i^{(l)} \gamma_j^{(l)} = \sum_k N_{ij}^k \gamma_k^{(l)}. \tag{10}$$

The general study of these fusion algebras and their classification have been the object of much work [8–10].

The numbers

$$d_i \doteq \gamma_i^{(0)} = \frac{S_{i0}}{S_{00}} \tag{11}$$

appear as statistical dimensions of superselection sectors [12, 13] in algebraic quantum field theory, as square roots of indices for inclusions of von Neumann algebras [14], as relative sizes of highest weight modules of chiral symmetry algebras in conformal field theory [4], and in connection with truncated tensor products of quantum groups (see [15] for an accomplished review). According to equation (10), these numbers obey the statistical dimension sum rules

$$d_i d_j = \sum_k N_{ij}^k d_k. \tag{12}$$

which shows that d_i is a Frobenius eigenvalue of N_i.

3. $SU(N)_2$ cominimal equivalence classes

At the level $K = 2$ the central charge of the chiral $SU(N)$ models is given by

$$c = \frac{2(N-1)}{N+2} \tag{13}$$

and their primary fields are identified with the order fields σ_k, $k = 0, 1, \ldots, N-1$; Z_N-neutral fields $\epsilon^{(j)}$, $j = 1, 2, \ldots \leqslant N/2$ and the parafermionic currents Ψ_k, $k = 1, \ldots, N-1$, in the Zamolodchikov–Fateev parafermionic theories [16]. For each primary field we define a 'charge' $\nu = 0, 1, \ldots, 2(N-1) \mod 2N$ and we collect the $N(N+1)/2$ primary fields in N *cominimal equivalence classes* [17], $[\phi_k^k]$, $k = 0, 1, \ldots, N-1$, according to their statistical dimensions:

$$d_k = \prod_{i=0}^{k-1} \frac{s(N-i)}{s(i+1)} \qquad s(x) = \sin\left(\frac{x\pi}{N+2}\right)$$

$$d_0 = 1 \qquad d_{N-k} = d_k \qquad k = 1, 2, \ldots, N-1. \tag{14}$$

$SU(N)_2$ representations of the order fields ϕ_k^k, $k = 1, \ldots, N-1$ are the fully antisymmetric Young tableaux with k boxes (i.e. the reduced tableau which is a column with k boxes). Tableaux of fields comprising a cominimal equivalence class ϕ_ν^k in which the representation ϕ_k^k appears, ($\nu = k \mod 2$, i.e. $\nu = k, k+2, \ldots, 2N-2-k$), are obtained by adding $(\nu - k)/2$ rows of width 2 to the top of the reduced tableau of ϕ_k^k. Therefore ϕ_ν^k is a Young tableau of two columns with ν boxes, since $(\nu + k)/2$ boxes in the first column and $(\nu - k)/2$ in the second column.

The conformal weights of the fields comprising a cominimal equivalence class in which the representation ϕ_k^k appears are simply related to the conformal weight of ϕ_k^k by

$$\Delta_\nu^k = \Delta_k^k + \frac{\nu - k}{4N}(2N - \nu - k) \tag{15}$$

and the conformal dimensions of the order fields [16] are given by

$$\Delta_k^k = \frac{k(N-k)}{2N(N+2)}. \tag{16}$$

These equivalence classes are generated by Z_N symmetry, connecting the representations belonging to each class through the fusion rules [18]:

$$\phi_{\nu_1}^{k_1} \times \phi_{\nu_2}^{k_2} = \sum_{k=|k_1-k_2| \mod 2}^{\min(k_1+k_2, 2N-k_1-k_2)} \phi_{\nu_1+\nu_2}^{k}. \tag{17}$$

In particular, the elementary field ϕ_1^1, ($\phi_1^1 \times \phi_\nu^k = \phi_{\nu+1}^{k-1} + \phi_{\nu+1}^{k+1}$) connects the equivalence class of ϕ_ν^k with adjacent classes, while the field ϕ_2^0, ($\phi_2^0 \times \phi_\nu^k = \phi_{\nu+2}^k$), connects the fields in the same cominimal equivalence class. Thus, the $SU(N)_2$ fusion ring can be generated by these two fields. For example, the 10 primary fields of $SU(4)_2$ can be collected in four cominimal equivalence classes as

$$\left\{ \begin{array}{ccccccccccc} & & & \phi_3^3 & & & & \to & d_3 = \frac{s(2)}{s(1)} \\ & & \nearrow & & \searrow & & & & \\ & \phi_2^2 & & & & \phi_4^2 & & \to & d_2 = \frac{s(3)}{s(1)} \\ \nearrow & & \searrow & & \nearrow & & \searrow & & \\ \phi_1^1 & & & \phi_3^1 & & & \phi_5^1 & \to & d_1 = \frac{s(4)}{s(1)} \\ \nearrow \;\; \searrow & & \nearrow & & \searrow & \nearrow & \;\; \searrow & & \\ \phi_0^0 \quad \phi_2^0 & & & \phi_4^0 & & & \phi_6^0 & \to & d_0 = \frac{s(5)}{s(1)} \end{array} \right\}. \tag{18}$$

These cominimal equivalence classes provide a representation of the Z_4 symmetry and the primary fields corresponding to representations in the same class differ only by free fields.

4. $SU(N)_2$ polynomial rings

Let us start by considering the case $SU(4)_2$ (the $SU(2)_2$ and $SU(3)_2$ cases were considered in [2]).

The variables x and y are associated with the fields ϕ_1^1 and ϕ_2^0, respectively. Using $\phi_0^0 = 1$, the fusion rules (17) (see equation (18)) give the expressions of the other fields:

$$\begin{array}{llll} \phi_0^0 = 1 & \phi_1^1 = x & \phi_2^2 = x^2 - y & \phi_3^3 = x^3y - 2xy \\ \phi_2^0 = y & \phi_3^1 = xy & \phi_4^2 = x^2y - y^2 & \\ \phi_4^0 = y^2 & \phi_5^1 = xy^2 & & \\ \phi_6^0 = y^3 & & & \end{array} \tag{19}$$

and from the identification $\phi_\nu^k = \phi_{4+\nu}^{4-k}$ mod 8 we obtain the following constraints:

$$\begin{array}{ll} x^4 - 3x^2y + y^2 = 1 & x^3y - 2xy^2 = x \\ x^2y^2 - y^3 = x^2 - y & x^3 - 2xy = xy^3 \\ y^4 = 1. & \end{array} \tag{20}$$

These constraints can be combined and reduced to a one-variable constraint

$$x^{10} - 8x^6 - 9x^2 = 0 \tag{21}$$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_1^1}$, and its eigenvalue 0 is doubly degenerate, implying that x may not be inverted on the ring. Similarly, one can eliminate x from (20) and obtain a one-variable constraint in y:

$$y^{10} - y^8 - 2y^6 + 2y^4 + y^2 - 1 = 0 \tag{22}$$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_2^0}$, whose eigenvalues are degenerate. Thus, the fusion ring of the $SU(4)_2$ model can be expressed in terms of two variables associated with the representations ϕ_1^1 and ϕ_2^0 which satisfy independent constraint equations.

Next, let us consider the 15 primary fields of the chiral $SU(5)_2$ model which can be collected in five cominimal equivalence classes as

$$\left\{\begin{array}{ccccccccccccccccccc}
& & & & & & & & \phi_4^4 & & & & & & & & & \to & d_4 = \frac{s(2)}{s(1)} \\
& & & & & & & \nearrow & & \searrow & & & & & & & & & \\
& & & & & & \phi_3^3 & & & & \phi_5^3 & & & & & & & \to & d_3 = \frac{s(3)}{s(1)} \\
& & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & & & & \\
& & & & \phi_2^2 & & & & \phi_4^2 & & & & \phi_6^2 & & & & & \to & d_2 = \frac{s(4)}{s(1)} \\
& & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & & \\
& & \phi_1^1 & & & & \phi_3^1 & & & & \phi_5^1 & & & & \phi_7^1 & & & \to & d_1 = \frac{s(5)}{s(1)} \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\
\phi_0^0 & & & & \phi_2^0 & & & & \phi_4^0 & & & & \phi_6^0 & & & & \phi_8^0 & \to & d_0 = \frac{s(6)}{s(1)}
\end{array}\right\}. \tag{23}$$

The variables x and y are associated with the fields ϕ_1^1 and ϕ_2^0, respectively. Using $\phi_0^0 = 1$, the fusion rules (17) give the expressions for the other fields:

$$\begin{array}{lll} \phi_0^0 = 1 & \phi_1^1 = x & \phi_4^2 = x^2y - y^2 \\ \phi_2^0 = y & \phi_3^1 = xy & \phi_6^2 = x^2y^2 - y^3 \\ \phi_4^0 = y^2 & \phi_5^1 = xy^2 & \phi_3^3 = x^3 - 2xy \\ \phi_6^0 = y^3 & \phi_7^1 = xy^3 & \phi_5^3 = x^3y - 2xy^2 \\ \phi_8^0 = y^4 & \phi_2^2 = x^2 - y & \phi_4^4 = x^4 - 3x^2y + y^2 \end{array} \tag{24}$$

and the identification $\phi_\nu^k = \phi_{5+\nu}^{5-k}$ mod 10 gives us the following constraint equations:

$$\begin{array}{ll} x^5 - 4x^3y + 3xy^2 = 1 & x^2y^3 - y^4 = x^3 - 2xy \\ x^4y - 3x^2y^2 + y^3 = x & x^4 - 3x^2y + y^2 = xy^4 \\ x^3y^2 - 2xy^3 = x^2 - y & y^5 = 1. \end{array} \tag{25}$$

These constraints can be combined and reduced to a one-variable constraint equation

$$x^{15} - 16x^{10} - 57x^5 + 1 = 0 \tag{26}$$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_1^1}$, whose eigenvalues are non-degenerate. It means that x may be inverted on the ring: we can eliminate y from the constraint equations (25) as

$$y = \tfrac{1}{181}(-14x^{12} + 221x^7 + 910x^2). \tag{27}$$

Substituting this value of y in (24) we will obtain a polynomial ring in a single variable:

$P_0^0(x) = 1$

$P_2^0(x) = \frac{1}{181}(910x^2 + 221x^7 - 14x^{12})$

$P_4^0(x) = \frac{1}{181}(4592x^4 + 1260x^9 - 79x^{14})$

$P_6^0(x) = \frac{1}{181}(404x + 155x^6 - 9x^{11})$

$P_8^0(x) = \frac{1}{181}(2043x^3 + 597x^8 - 37x^{13})$

$P_2^2(x) = -\frac{1}{181}(729x^2 + 221x^7 - 14x^{12})$

$P_4^2(x) = -\frac{1}{181}(3682x^4 + 1039x^9 - 65x^{14})$

$P_6^2(x) = -\frac{1}{181}(325x + 66x^6 - 5x^{11})$.

$P_1^1(x) = x$

$P_3^1(x) = \frac{1}{181}(910x^3 + 221x^8 - 14x^{13})$

$P_5^1(x) = \frac{1}{181}(79 + 89x^5 - 4x^{10})$

$P_7^1(x) = \frac{1}{181}(404x^2 + 155x^7 - 9x^{12})$

$P_3^3(x) = -\frac{1}{181}(1639x^3 + 442x^8 - 28x^{13})$

$P_5^3(x) = -\frac{1}{181}(144 + 66x^5 - 5x^{10})$

$P_4^4(x) = \frac{1}{181}(2043x^4 + 597x^9 - 37x^{14})$

These $P_\nu^k(x)$ polynomials define a (modulo $x^{15} - 16x^{10} - 57x^5 + 1$) one-variable $SU(5)_2$ polynomial ring.

Similarly, one can eliminate x from (25) and obtain a one-variable constraint in y:

$$y^{15} - 3y^{10} + 3y^5 - 1 = 0 \tag{28}$$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_2^0}$, but their eigenvalues are degenerate.

We now extend this construction to the whole set of $SU(N)_2$ models. We associate the following polynomials with each irreducible representation ϕ_ν^k:

$$P_\nu^k(x, y) = \sum_{n=0}^{[k/2]} (-1)^n \frac{(k-n)!}{n!(k-2n)!} x^{k-2n} y^{n+(\upsilon-k)/2} \tag{29}$$

where $k = 0, 1, \ldots, N-1$, $\nu = k \mod 2$, i.e. $\nu = k, k+2, \ldots, 2(N-1) - k$ and $[k/2]$ means the largest integer less than or equal to $k/2$.

The identification $\phi_\nu^k = \phi_{N+\nu}^{N-k} \mod 2N$ gives the corresponding one-variable constraint equations:

$$x^{N/2} \prod_{n=1}^{N/2} (x^N + (-1)^n d^N(n)) = 0 \qquad (y^{N/2} - 1)^{(N+2)/2} (y^{N/2} + 1)^{N/2} = 0 \tag{30}$$

for the cases when N is even, and

$$\prod_{n=1}^{(N+1)/2} (x^N - d^N(n)) = 0 \qquad (y^N - 1)^{(N+1)/2} = 0 \tag{31}$$

for the cases when N is odd. In these expressions we have introduced the numbers

$$d(n) = \frac{\sin(n\pi N/(N+2))}{\sin(n\pi/(N+2))} \qquad n = 1, 2, \ldots, \leqslant \frac{N+2}{2}. \tag{32}$$

Inspecting the constraint equations in the variable y we can see that the fusion matrices $N_{\phi_2^0}$ are degenerate for all $SU(N)_2$ models. It means that we cannot eliminate the variable x

from the polynomials (29). If N is even and $N > 2$, we can see from equations (30) that of the eigenvalues of the fusion matrices $N_{\phi_1^1}$ only zero is degenerate ($N/2$ times), following that x also cannot be inverted on these rings. It means that we also cannot eliminate the variable y from (29) and the corresponding fusion ring is represented by a polynomial ring in two variables.

On the other hand, if N is odd or $N = 2$, the eigenvalues of the fusion matrices $N_{\phi_1^1}$ are not degenerate and x may be inverted on the ring. We can therefore solve for y as a function of x using the corresponding constraint equations which were reduced to (31), and the fusion ring is faithfully represented by polynomials in one variable. For instance, the next odd-N model is $SU(7)_2$ for which the constraint equation is $x^{28} - 64x^{21} - 157x^{14} + 1640x^7 + 1 = 0$ and it is possible eliminate y from (29) using

$$y = \frac{1}{66\,4276}(2958x^{23} - 189\,549x^{16} - 4\,653\,716x^9 + 5\,504\,583x^2) \tag{33}$$

and we obtain the resulting fusion ring as a polynomial ring in one variable.

At this point we can proceed to the generalization of these results by explicit diagonalization of fusion matrices of the chiral $SU(N)_2$ models. With each irreducible representation ϕ_ν^k we associate a factored characteristic equation $\det(x\mathbb{1} - N_{\phi_\nu^k}) = 0$ which depend on the parafermionic charge ν according to $N = (p/q)\nu$, where p and q are mutually coprime positive integers

$$\prod_{n=1}^{(N+1)/2} \left(x^p - d_k^p(n)\right)^{\nu/q} = 0 \qquad \text{if } p.q \text{ odd} \tag{34}$$

$$\prod_{n=1}^{(N+1)/2} \left(x^p + (-1)^n d_k^p(n)\right)^{\nu/q} = 0 \qquad \text{if } p.q \text{ even} \tag{35}$$

for N odd, and

$$\left(x^p - d_k^p(l)\right)^{\nu/2q} \prod_{n=1}^{N/2} \left(x^p - d_k^p(n)\right)^{\nu/q} = 0 \qquad \text{if } p.q \text{ odd} \tag{36}$$

$$\left(x^p + (-1)^l d_k^p(l)\right)^{\nu/2q} \prod_{n=1}^{N/2} \left(x^p + (-1)^n d_k^p(n)\right)^{\nu/q} = 0 \qquad \text{if } p.q \text{ even} \tag{37}$$

where $l = (N+2)/2$, for N even.

Here we have introduced a generalization of the numbers $d(n)$ of equation (32):

$$d_k(n) = \frac{\sin(n(N+1-k)\pi/(N+2))}{\sin(n\pi/(N+2))}$$

$$k = 0, 1, 2, \ldots, N-1 \quad n = 1, 2, \ldots, \leqslant \frac{N+2}{2} \tag{38}$$

which satisfy the following sum rules:

$$d_i(n)d_j(n) = \sum_k (N_i)_j^k d_k(n). \tag{39}$$

From these numbers we observe that the characteristic polynomials of the fusion matrices of the fields comparing the same cominimal equivalence classes have equivalent spectra of zeros, i.e. they differ only in the Z_N-degeneracy of their eigenvalues which depend on of the parafermionic charge through the relation $N = p\nu/q$.

Therefore there are many alternative ways of constructing the $SU(N)_2$ polynomial rings in two-variables: take for y any field belonging to any equivalence class $[\phi_k^k]$. The fusion rules (17) give us four possibilities (at most) from which to choose the field associated with the variable x. The corresponding constraint equations are given by equations (34)–(37). If at least one of the fusion matrices associated with x and y is non-degenerate, it is possible to eliminate one of variables, resulting in a polynomial ring in a single variable.

These results tell us that for N odd $SU(N)$ possess a single-variable polynomial ring at level $K = 2$. For other values of K, as observed by Gannon [19], $SU(2)$ and $SU(3)$ are the only $SU(N)$ whose fusion rings at all levels K can be represented by polynomials in only one variable. For each $N > 3$, there will be infinitely many K for which the fusion ring $SU(N)_K$ requires more than one variable, and infinitely many other K for which one variable will suffice.

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